

Landau problem with a general time-dependent electric field

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Abstract

The time evolution is studied for the Landau level problem with a general time dependent electric field $\mathbf{E}(t)$ in a plane perpendicular to the magnetic field. A general and explicit factorization of the time evolution operator is obtained with each factor having a clear physical interpretation. The factorization consists of a geometric factor (path-ordered magnetic translation), a dynamical factor generated by the usual time-independent Landau Hamiltonian, and a nonadiabatic factor that determines the transition probabilities among the Landau levels. Since the path-ordered magnetic translation and the nonadiabatic factor are, up to completely determined numerical phase factors, just ordinary exponentials whose exponents are explicitly expressible in terms of the canonical variables, all of the factors in the factorization are explicitly constructed. The numerical phase factors are quantum mechanical in nature and could be of significance in interference experiments. The factorization is unique from the point of view of the quantum adiabatic theorem and provides a demonstration of how the quantum adiabatic theorem (incorporating the Berry phase phenomenon) is realized when infinitely degenerate energy levels are involved. Since the factorization separates the effect caused by the electric field into a geometric factor and a nonadiabatic factor, it makes possible to calculate the nonadiabatic transition probabilities near the adiabatic limit. A formula for matrix elements that determines the mixing of the Landau levels for a general, non-adiabatic evolution is also provided by the factorization.

1 Introduction

The magnetic translation concept is much discussed in the literature [1, 2, 3, 4, 5, 6, 7]. The basic example is the quantum mechanics of a charged particle moving in a two-dimensional plane perpendicular to a magnetic field, i.e., the Landau level problem. The Hamiltonian does not commute with the usual translation operator generated by the canonical momentum. The translation symmetry of the physical situation is realized through magnetic translation. However, in this simple context, the solution of the problem (energy levels, etc.) does not seem to rely on this concept in any essential way. The magnetic translation concept has been used [8, 9, 10, 11] in condensed matter physics including Bloch electrons in a magnetic field where there is a lattice potential present. Yet we believe the essential role of magnetic translation in exploring time-dependent problems have yet to be fully explored.

In this paper, we study the quantum mechanical problem of a charged particle moving in a two-dimensional plane subjected to a uniform magnetic field perpendicular to the plane and a spatially uniform but time-dependent in-plane electric field $\mathbf{E}(t)$. The Hamiltonian for such a system is the sum of the usual Landau level Hamiltonian H_0 and the potential energy of the charge in the electric field:

$$\begin{aligned} H &= H_0 + V(\mathbf{x}, t), \\ &= \frac{1}{2m} \left[\mathbf{p} - \frac{q}{c} \mathbf{A}(\mathbf{x}) \right]^2 - qE_1(t)x_1 - qE_2(t)x_2, \end{aligned} \quad (1)$$

where $\mathbf{A}(\mathbf{x})$ satisfies $\nabla \times \mathbf{A}(\mathbf{x}) = B\mathbf{e}_3$. The situation contains the usual Landau level problem as a special case, where $E_1(t) = E_2(t) = 0$. It also contains the special case for which the uniform electric field is constant in time, and the case [12] for which $\mathbf{E}(t)$ depends on time but is along a fixed direction. These cases are all known in the literature. Our purpose here is to study the general case with $\mathbf{E}(t) = E_1(t)\mathbf{e}_1 + E_2(t)\mathbf{e}_2$ being a general time-dependent electric field. This general situation cannot be reduced to the sum of two special cases and it will be shown that it has new features of its own. In particular, we will show that a path-ordered magnetic translation, which is an element of the magnetic translation group, plays an essential role in describing the evolution of the system. This path-ordered magnetic translation contains a numerical phase factor that is nontrivial only when the electric field changes direction with time. Our main result is a factorization of the time-evolution operator into three factors, each having a clear physical meaning. The method and result seem to be quite natural that it is possible that they may find applications in studying more complicated situations such as the Landau problem on a cylinder [13] or a torus, or when there is a lattice potential present.

First of all, from the physical picture that underlies the Hall effect, the circular orbit of a charged particle in mutually perpendicular electric and magnetic fields exhibits motion in the direction of $\mathbf{E} \times \mathbf{B}$. It is easy to conjecture that, in the case where $\mathbf{E}(t)$ is a general time-dependent electric field, the instantaneous velocity of the global motion of the circular orbit is $cE(t)/B$, in the direction of $\mathbf{E} \times \mathbf{B}$. Therefore, the position of the center of the circular orbit at time t is described by a parameter

$$\mathbf{R}(t) = \frac{c}{B} \int_0^t (E_2(s)\mathbf{e}_1 - E_1(s)\mathbf{e}_2) ds, \quad (2)$$

where we have taken the initial position $\mathbf{R}(0)$ to be at the origin of the parameter space: $\mathbf{R}(0) = \mathbf{0}$.

In this paper, we aim to find a general factorization of the time-evolution operator $U(t, 0)$ corresponding to H in terms of three factors: a geometrical factor describing the physical displacement of a quantum wave by the amount of $\mathbf{R}(t)$, a dynamical factor generated by the usual time-independent Landau Hamiltonian H_0 , and another factor that describes the mixing of the Landau levels of H_0 . The factorization is general in the sense that it is valid for a general time variation of the function $\mathbf{R}(t)$ and therefore of $\mathbf{E}(t)$.

We comment here that previous work on the special cases, such as the one in [12] has not relied on the magnetic translation concept. This is due to the fact that if the electric field is along a fixed direction, say in the x_1 direction, one chooses the Landau gauge where $\mathbf{A}(\mathbf{x}) = (0, Bx, 0)$ and furthermore specify the constant of motion p_2 to be 0 and the problem is then simplified. Such a simplification also implies that the

quantum states under consideration have $p_2 = 0$. For such a restriction, the geometric operator found in the present paper is equal to the identity. So the only effects caused by the electric field are nonadiabatic transitions. No such simplification exists if the electric field changes direction with time. In order to treat the general case, the magnetic translation concept is necessary. In fact, even in the special case, if one does not make the special restrictions, there should be a magnetic translation along the fixed direction of x_2 . We also note that upon choosing a specific gauge, the Hamiltonian considered here can be cast in a quadratic form. For such type of time-dependent Hamiltonians, there is a method that uses the corresponding classical solutions to construct the propagator of the quantum problem [14, 15]. While such an approach could be useful as a general theory, it does not seem to point to a factorization of the time evolution operator which, in specific contexts, can make the time evolution transparent.

2 Factorization of the time-evolution operator

In this section, we first give a discussion on the properties of the variables (π_1, π_2) and (w_1, w_2) which can be expressed in terms of the canonical variables \mathbf{x} and \mathbf{p} . Then we use these variables to construct a factorization of the time evolution operator. Although the commutation relations among these variables hold in any picture, we assume throughout this paper that when these variables appear in the exponents, they are Schrödinger picture variables.

In the usual Landau problem H_0 , the kinematical momentum is $\pi_\mu = p_\mu - \frac{q}{c}A_\mu(\mathbf{x})$, $\mu = 1, 2$, where $A_\mu(\mathbf{x})$ is the vector potential in arbitrary gauge, though for simplicity we assume that $A_\mu(\mathbf{x})$ does not depend on time. Define

$$w_1 = \pi_1 - \frac{qB}{c}x_2 = -\frac{qB}{c}c_2, \quad w_2 = \pi_2 + \frac{qB}{c}x_1 = \frac{qB}{c}c_1. \quad (3)$$

In classical mechanics, (c_1, c_2) is the center of the circular motion of the charged particle in the magnetic field. In quantum mechanics, because of the canonical commutation relations, c_1 and c_2 do not commute. We have the following commutation relations

$$[\pi_1, \pi_2] = i\hbar qB/c, \quad [w_1, w_2] = -i\hbar qB/c, \quad [\pi_\mu, w_\nu] = 0. \quad (4)$$

To realize a translation $\mathbf{x} \rightarrow \mathbf{x} + \mathbf{R}(t)$, where $\mathbf{R}(t)$ traverses through a path $C_{\mathbf{R}}$ in parameter space, one may use the ordinary translation operator $\exp(-ip_\mu R_\mu(t)/\hbar)$. (Summation over repeated indices is assumed.) However, the ordinary translation does not commute with π_μ . The magnetic translation operator, defined as

$$\overline{M}(\mathbf{R}) = \exp(-iw_\mu R_\mu/\hbar), \quad (5)$$

is a generalization of the ordinary translation operator when a magnetic field is present: It physically translates a quantum wave because it commutes with the kinematical momentum, and in the simplest Landau system H_0 , it preserves the energy. Note that unlike ordinary translation, a distinction between $\exp(-iw_\mu R_\mu/\hbar)$ and the path-ordered magnetic translation

$$M(C_{\mathbf{R}}) = P \exp(-i\hbar^{-1}w_\mu \int_{C_{\mathbf{R}}} dR_\mu) = P \exp(-i\hbar^{-1}w_\mu R_\mu) \quad (6)$$

has to be made, for a general path $C_{\mathbf{R}}$ that is not in a straight line, because w_1 and w_2 do not commute. (We assume $R_\mu(0) = 0$.) Because of the simple commutation relation $[w_1, w_2] = -i\hbar qB/c$, their difference is a numerical phase factor, i.e.

$$M(C_{\mathbf{R}}) = e^{i\beta(C_{\mathbf{R}})} \exp(-iw_\mu R_\mu/\hbar), \quad (7)$$

where $\beta(C_{\mathbf{R}})$ is a real number determined by the path $C_{\mathbf{R}}$ traversed by $\mathbf{R}(t)$. In particular, for a closed path, $\beta(C_{\mathbf{R}})$ is equal to $-\frac{q\phi}{\hbar c}$, where ϕ is the magnetic flux enclosed by the loop C . This follows from the definition of a path-ordered exponential, the commutation relation $[w_1, w_2] = -i\hbar qB/c$ and the formula $e^A e^B = e^{A+B} e^{\frac{1}{2}[A, B]}$ for A and B commuting with $[A, B]$. For an open path $\mathbf{R}(t)$, the flux is through the

area enclosed by the path and the straight line pointing from the end point $\mathbf{R}(t)$ to the initial point $\mathbf{R}(0) = \mathbf{0}$. In general we have

$$\beta(C_{\mathbf{R}}) = -\frac{qB}{\hbar c} \mathbf{e}_3 \cdot \frac{1}{2} \int_{C_{\mathbf{R}}} \mathbf{R} \times d\mathbf{R} = -\frac{qB}{\hbar c} S. \quad (8)$$

Or, if we denote $R(t) = R_1(t) + iR_2(t)$, then

$$\beta(C_{\mathbf{R}}) = -\frac{qB}{\hbar c} \frac{(-i)}{4} \int_{C_{\mathbf{R}}} (R^* dR - R dR^*). \quad (9)$$

Another path-ordered exponential that is relevant to our purpose is

$$J(C_u) = P \exp \left(\pi u / \hbar - \pi^\dagger u^* / \hbar \right), \quad (\pi = \pi_1 + i\pi_2) \quad (10)$$

where the path C_u is the one traversed by $u(t)$ in a complex u -plane. Similar to the path-ordered magnetic translation, this path-ordered exponential can be evaluated by using the formula $[\pi, \pi^\dagger] = 2\hbar qB/c$. We have

$$J(C_u) = e^{i\gamma(C_u)} \exp \left(\pi u / \hbar - \pi^\dagger u^* / \hbar \right), \quad (11)$$

where

$$\gamma(C_u) = i \frac{qB}{\hbar c} \int_{C_u} (u^* du - u du^*) = -\frac{qB}{\hbar c} 4S(C_u). \quad (12)$$

$S(C_u)$ is the area enclosed by the path traversed by $u(t)$ in the complex u -plane and the straight line connecting the end and initial points of the path.

The basic idea that leads to the factorization of $U(t, 0)$ is to switch from the Schrödinger picture to the Heisenberg picture first. Furthermore, we observe that the Heisenberg equations of motion decouple from each other for the variables $\pi_\mu(t)$ and $w_\mu(t)$, unlike the equations of motion for the canonical variables. It is the behavior of $\pi_\mu(t)$ and $w_\mu(t)$ instead of that of $p_\mu(t)$ and $x_\mu(t)$ that leads to the factorization of $U(t, 0)$; namely we find an operator $O(t, 0)$ in a factorized form that recovers the evolution of $\pi_\mu(t)$ and $w_\mu(t)$ through $\pi_\mu(t) = O^\dagger(t, 0) \pi_\mu O(t, 0)$ and $w_\mu(t) = O^\dagger(t, 0) w_\mu O(t, 0)$. We then verify that this operator is in fact $U(t, 0)$ by a computation that shows it satisfies the Schrödinger equation with the initial condition $U(0, 0) = I$. These steps are presented in detail in Appendix A. The exact and general factorization of the time evolution operator $U(t, 0)$ is then determined to be

$$U(t, 0) = M(C_{\mathbf{R}}) D(t) J(C_u), \quad (13)$$

where $D(t) = \exp(-iH_0 t / \hbar)$ is the evolution generated by the usual Landau Hamiltonian; $M(C_{\mathbf{R}})$ describes path-ordered magnetic translation of a wave corresponding to the path $C_{\mathbf{R}}$ traversed by $\mathbf{R}(t) = \frac{c}{B} \int_0^t (E_2(s) \mathbf{e}_1 - E_1(s) \mathbf{e}_2) ds$. These two operators commute with each other, so no energy is gained or lost by the action on the wave-function of a magnetic translation. The complex parameter u that determines the operator $J(C_u)$ is given by

$$u(t) = \frac{i}{2} \int_0^t e^{-i\omega s} \frac{d}{ds} R^*(s) ds = \frac{-c}{2B} \int_0^t e^{-i\omega s} E^*(s) ds, \quad (14)$$

where $R^*(s) = R_1(s) - iR_2(s)$, $E^*(s) = E_1(s) - iE_2(s)$, and $\omega = qB/(mc)$.

We see that the operator $\exp(\pi u / \hbar - \pi^\dagger u^* / \hbar)$ in $J(C_u)$ mixes different energy levels of H_0 . This is due to the fact that π^\dagger and π have the meaning of being proportional to the creation and annihilation operators: the Hamiltonian H_0 can be written as $H_0 = \hbar\omega(a^\dagger a + 1/2)$, where $a^\dagger = (\hbar k)^{-1} \pi^\dagger$, $a = (\hbar k)^{-1} \pi$, with $(\hbar k)^{-1} = \sqrt{c/(2qB\hbar)}$. From this we see that the operator $J(C_u)$ has the meaning that, when acting on any of the ground states of H_0 , it generates a coherent state associated with the minimization of $\Delta\pi_1 \cdot \Delta\pi_2$ rather than the minimization of $\Delta x_1 \Delta p_1$, or of $\Delta x_2 \Delta p_2$. The coherent state nature of $J(C_u)|\Psi_0\rangle$, where $|\Psi_0\rangle$ is a ground state of H_0 , is then preserved under the action of $M(C_{\mathbf{R}})D(t)$ in the time-evolution operator.

3 Gauge invariance and explicitness of the factorization

Apart from the numerical phase factors $e^{i\beta(C_{\mathbf{R}})}$ and $e^{i\beta(C_u)}$, the operators $M(C_{\mathbf{R}})$ and $J(C_u)$ are generated by $(w_1, w_2) = (-\frac{qB}{c}c_2, \frac{qB}{c}c_1)$ and (π_1, π_2) respectively. In the system H_0 , the physical meaning of these generators are given by the “center of the circular orbit” and the kinematical momentum which are gauge independent. However, one must be careful in claiming the factorization gauge invariant in the most general way since the derivation of the factorization relies on the condition that π_μ and w_μ in the exponents are time-independent Schrödinger variables and as such, A_μ is assumed to depend on \mathbf{x} only. For example, in the symmetric gauge where $\mathbf{A}(\mathbf{x}) = \frac{B}{2}\mathbf{e}_3 \times \mathbf{x}$, we have $\pi_1 = p_1 + \frac{qB}{2c}x_2$, $\pi_2 = p_2 - \frac{qB}{2c}x_1$, $w_1 = p_1 - \frac{qB}{2c}x_2$ and $w_2 = p_2 + \frac{qB}{2c}x_1$. Therefore, the three factors in the factorization are all explicit functions of the canonical variables once a specific gauge is chosen.

Consider the example of a rotating electric field. We have $R(t) = R_0(e^{-i\nu t} - 1)$ or $E(t) = (iB/c)\dot{R}(t) = (\nu B/c)R_0 e^{-i\nu t} = E_0 e^{-i\nu t}$. By the formulas (9), (12) and (14), we have $\beta(C_{\mathbf{R}}) = (qB/\hbar c)\frac{1}{2}R_0^2(\nu t - \sin \nu t)$, $u(t) = (-R_0\nu/2)(e^{-i\omega t + i\nu t} - 1)/(-i\omega + i\nu)$, and $\gamma(C_u) = (qB/\hbar c)(R_0^2/2)(\frac{\nu}{\omega - \nu})^2[(\omega - \nu)t - \sin(\omega - \nu)t]$. For the case of $\nu \rightarrow \omega$, we have $u(t) = -R_0\nu t/2$, and $\gamma(C_u) = 0$. The expressions for $M(C_{\mathbf{R}})$ and $J(C_u)$ are therefore obtained according to (7) and (11). In this particular example, the result implies that resonance happens at $\nu = \omega$ where $u(t)$ increases linearly with time, thereby causing rapid transitions among the Landau levels. This example will be further commented on at the end of the paper.

4 Physical implications of the factorization

The general factorization of $U(t, 0)$ demonstrates that a path-ordered magnetic translation is a natural concept associated with a charged particle in a time-dependent electric field and a uniform perpendicular magnetic field. The same can be said of the path ordered exponential $J(C_u)$ that describes the mixing of the Landau levels. The factorization also has a natural connection with the quantum adiabatic theorem as discussed in the next section. The numerical phase factors $e^{i\gamma(C_u)}$ in $J(C_u)$ and $e^{i\beta(C_{\mathbf{R}})}$ in $M(C_{\mathbf{R}})$, which distinguish the path-ordered exponentials from the direct exponentials, represent pure quantum effects that have no classical origin and they could be of consequences in interference experiments. It is obvious that the factor $e^{i\beta(C_{\mathbf{R}})}$ is nontrivial only when the electric field changes direction with time. This numerical factor contains the adiabatic Berry phase in the usual sense as discussed in the next section. One may want to draw an analogy with the Aharonov-Bohm phase even though the magnetic field interacts directly with the particle here. However, the phase $\beta(C_{\mathbf{R}}) = -\frac{q\phi}{\hbar c}$ that results from a path-ordered magnetic translation around a closed path $C_{\mathbf{R}}$ has an opposite sign from the Aharonov-Bohm phase. The derivation of the latter from the point of view of the geometric phase is given in Berry’s paper [16].

5 Relation to the quantum adiabatic theorem

The factorization can be viewed from a different though equivalent perspective. The Hamiltonian H can be recast in the form

$$H_L = \frac{1}{2m} \left[\mathbf{p} - \frac{q}{c} \mathbf{A}_L(x, \mathbf{R}) \right]^2, \quad (15)$$

through the gauge transformation

$$\Psi_L(\mathbf{x}, t) = \exp\left[-i\frac{q}{\hbar c}\chi(\mathbf{x}, \mathbf{R})\right]\Psi(\mathbf{x}, t), \quad (16)$$

$$\mathbf{A}_L(x, \mathbf{R}) = \mathbf{A}(\mathbf{x}) - \nabla\chi(\mathbf{x}, \mathbf{R}) = \mathbf{A}(\mathbf{x}) - B\mathbf{e}_3 \times \mathbf{R}(t), \quad (17)$$

where

$$\chi(\mathbf{x}, t) = -BR_2(t)x_1 + BR_1(t)x_2. \quad (18)$$

Because $\Psi_L(\mathbf{x}, 0) = \Psi(\mathbf{x}, 0)$, the time-evolution operators $U_L(t, 0)$ and $U(t, 0)$ are related by

$$U_L(t, 0) = \exp[-i \frac{q}{\hbar c} \chi(\mathbf{x}, \mathbf{R})] U(t, 0). \quad (19)$$

The Landau level Hamiltonian $H_L(t)$, unlike the gauge equivalent $H(t)$, has energy eigenvalues $E_n = \hbar\omega(n + 1/2)$, where $\omega = qB/(mc)$, that depend on B only and is independent of the time variation of $H_L(t)$. Therefore, it is in the gauge of $H_L(t)$ that $D(t)$ carries the dynamical phase factor of adiabatic evolutions of eigenstates of the Hamiltonian. This should relate to the quantum adiabatic theorem, where $G(C_{\mathbf{R}}) = \exp[-i \frac{q}{\hbar c} \chi(\mathbf{x}, \mathbf{R})] M(C_{\mathbf{R}})$ is a geometrical operator completely determined by the path of $\mathbf{R}(t)$ which brings an initial eigenstate of $H_L(0)$ to an instantaneous eigenstate of $H_L(t)$, and $J(C_u)$ describes nonadiabatic transitions. This is an example of the quantum adiabatic theorem [17] where all of the three factors of the time-evolution operator are explicitly constructed and where the energy levels of the instantaneous Hamiltonian are infinitely degenerate.

Recall that the usual quantum adiabatic theorem (and also generalizations [18]) incorporating the Berry phase phenomenon is essentially a factorization of the time-evolution operator into three pieces: a geometric factor that embodies a Berry phase, a usual dynamical factor, and a nonadiabatic factor that approaches the identity operator in the adiabatic limit [17]. In our case, once a specific gauge is chosen, the factors $G(C_{\mathbf{R}}) = \exp[-i \frac{q}{\hbar c} \chi(\mathbf{x}, \mathbf{R})] M(C_{\mathbf{R}})$ and $J(C_u)$ are exponentials of explicit functions of the canonical variables, it thus provides an explicit example of the quantum adiabatic theorem involving infinitely degenerate energy levels.

In a previous work [5], we obtained a factorization of the time evolution operator for a charged particle in a slowly rotating magnetic field with a strong confining potential confining the particle to be in the plane that is perpendicular to the instantaneous magnetic field. There, the factorization is valid in the adiabatic limit only; i.e., no information about the nonadiabatic factor was obtained due to the complexity of the problem. The method adopted in the present paper may be applied to study the nonadiabatic factor in that situation.

6 Adiabatic perturbations

It is clear that because of the existence of the factor $M(C_{\mathbf{R}})$ in the time-evolution operator $U(t, 0)$, which gives rise to the geometric phase phenomenon, one cannot choose a fixed basis of eigenfunctions of H_0 , and perform a standard textbook version perturbative calculation. This is true even if the electric field is small, because it is the accumulative effects, i.e., \mathbf{R} in $M(C_{\mathbf{R}})$, that determines $M(C_{\mathbf{R}})$ which is not close to identity even if \mathbf{E} is small.

The factorization of $U(t, 0)$ allows us to do this perturbative calculation precisely because it identifies a set of parameter-dependent bases with the help of $M(C_{\mathbf{R}})$, then nonadiabatic transition probabilities are completely determined by $J(C_u)$.

The expression for $J(C_u)$ therefore allows the explicit calculation of non-adiabatic transition probabilities. Take an initial eigenstate $|\Psi(n, \mathbf{R}(0))\rangle = |\Psi(n, \mathbf{0})\rangle$ of the Hamiltonian $H_L(0)$ with eigenvalue $E_n = \hbar\omega(n + 1/2)$. The factorization of U_L says that the time-evolution of $|\Psi(n, \mathbf{0})\rangle$ can be seen as the action of $G(C_{\mathbf{R}})D(t)$ on top of $J(C_u)|\Phi(n, \mathbf{0})\rangle$. Since $G(C_{\mathbf{R}})D(t)$ does not cause transitions, the transition is caused by the action of $J(C_u)$ on $|\Phi(n, \mathbf{0})\rangle$ only. In the adiabatic limit where $|\dot{\mathbf{R}}(t)| \sim u(t)$ is small, one can expand $J(C_u)|\Phi(n, \mathbf{0})\rangle$ into

$$\begin{aligned} J(C_u)|\Phi(n, \mathbf{0})\rangle &= e^{i\gamma(C_u)}(1 + (\pi u/\hbar - \pi^\dagger u^*/\hbar)|\Phi(n, \mathbf{0})\rangle) + O(u^2), \\ &= e^{i\gamma(C_u)}(1 + (uka - ku^*a^\dagger)|\Phi(n, \mathbf{0})\rangle) + O(u^2). \end{aligned}$$

The transition probabilities are

$$\begin{aligned} |\langle \Phi(n-1, \mathbf{0}) | J(C_u) | \Phi(n, \mathbf{0}) \rangle|^2 &= |\langle \Phi(n-1, \mathbf{0}) | uka | \Phi(n, \mathbf{0}) \rangle|^2 + O(u^3), \\ &= nk^2 |u(t)|^2 + O(u^3), \end{aligned}$$

$$\begin{aligned}
|\langle \Phi(n+1, \mathbf{0}) | J(C_u) | \Phi(n, \mathbf{0}) \rangle|^2 &= |\langle \Phi(n+1, \mathbf{0}) | a^\dagger u^* / (\hbar k) | \Phi(n, \mathbf{0}) \rangle|^2 + O(u^3), \\
&= (n+1) \left(\frac{2qB}{\hbar c} \right) |u(t)|^2 + O(u^3).
\end{aligned}$$

All other transition probabilities are zero to the order of $O(u^2)$. So we see in this case that non-adiabatic transition probabilities are dependent on the energy levels, and increase with n . This is a result that shows explicitly how the quantum adiabatic theorem can be realized when an infinitely degenerate energy level is involved. As usual, the role of the oscillating $e^{-i\omega s}$ in the expression for $u(t)$ is to make the effect of $E^*(s)$ not to accumulate during an adiabatic process with $t \in [0, T]$, where T is a parameter that \mathbf{R} and \mathbf{E} may depend on through $\mathbf{R}(\frac{t}{T})$ and $\mathbf{E}(\frac{t}{T})$. If E_0/B is used as the dimensionless small parameter, say the electric field has a constant magnitude and that \mathbf{E} changes direction slowly compared with ω , then T can be chosen to be $(B/E)\omega^{-1}$. When $T \gg \omega^{-1}$, one estimates that $|u(t)| \leq \sim cE/B\omega$, for $t \in [0, T]$. The transition probabilities are then bounded by $n(\omega^{-1}c/l_B)^2 \cdot (E^2/B^2)$ for $t \in [0, T]$, where $l_B = \sqrt{\hbar c/(qB)}$ is the magnetic length. For an electron in a 15 T magnetic field, if $E = 1000$ volts/m, we have $E/B = 1000/(15 \times 3 \times 10^8)$, then $n(\omega^{-1}c/l_B)^2 \cdot (E^2/B^2) = n \cdot 1.45 \times 10^{-5}$, for the duration of $T = 1.71 \times 10^{-3}$ s. Then for a ground state for example, excitations can be expected to occur in about 100 s.

7 Mixing of the Landau levels beyond the adiabatic limit

One may also study the mixing of the Landau levels for a general nonadiabatic time evolution. Since $M(C_{\mathbf{R}})D(t)$ in the time evolution operator does not cause transitions among Landau levels of H_0 , the transitions are caused by the action of $J(C_u)$ on $|\Phi(n, \mathbf{0})\rangle$ only. In the most general case, the expression of $|\langle \Phi(m, \mathbf{0}) | J(C_u(t)) | \Phi(n, \mathbf{0}) \rangle|^2$ needs to be evaluated in order to determine the transition probability from an initial state (at $t = 0$) that is at the n -th Landau energy level to an m -th energy level at time t . We have

$$\begin{aligned}
J(C_u) &= e^{i\gamma(C_u)} \exp(\pi u / \hbar - \pi^\dagger u^* / \hbar), \\
&= e^{i\gamma(C_u)} \exp(uka - u^* ka^\dagger).
\end{aligned}$$

Using the formula $e^{A+B} = e^A e^B e^{-\frac{1}{2}[A, B]}$, and the commutation relation $[a, a^\dagger] = 1$, we have

$$J(C_u) = e^{i\gamma(C_u)} e^{-\frac{1}{2}|uk|^2} e^{-u^* ka^\dagger} e^{uka}.$$

Therefore, we have the following general expression for the matrix elements:

$$\langle m | J(C_u(t)) | n \rangle = e^{i\gamma(C_u)} e^{-\frac{1}{2}|uk|^2} (e^{-uka} | m \rangle)^\dagger (e^{uka} | n \rangle), \quad (20)$$

which represents mixing of the Landau energy levels for the most general type of the electric field which is not necessarily small and which does not have to change slowly.

In particular, this formula implies that for the ground state we have the following matrix element:

$$\langle 0 | J(C_u(t)) | 0 \rangle = e^{i\gamma(C_u)} e^{-\frac{1}{2}|uk|^2},$$

which implies that if we start with a ground state of H_0 at time 0, the probability that it remains to be in a ground state is $|\langle 0 | J(C_u(t)) | 0 \rangle|^2 = e^{-|uk|^2} = e^{-|u|^2 \frac{2qB}{\hbar c}}$. This implies, for the rotating electric field example as we discussed earlier where $E = E_0 e^{-i\omega t}$ and $u(t) = -cE_0 t / (2B)$ at the resonance of $\nu = \omega$, the probability for the state to remain in a ground state of H_0 (not necessarily the initial ground state because of the existence of the magnetic translation) is exactly

$$P_r(0 \rightarrow 0) = \exp[-2(E_0/B)^2 (c^2 t^2 / l_B^2)], \quad (21)$$

where $l_B = \sqrt{\hbar c / qB}$ is the magnetic length. (Note that for the electron, $q = -e$, the resonant electric field's angular velocity is then along the positive z direction, as expected. The transition therefore can take place extremely fast.)

That the resonance effect exists should be expected from the physical intuition gained from the classical solution. For the special case of an electric field along a fixed direction, say a sinusoidal electric field along the direction of x_1 , the transition probabilities can be calculated by choosing the gauge $\mathbf{A}(\mathbf{x}) = (0, Bx, 0)$ and the condition $p_2 = 0$ [12] which in effect makes the geometric operator equal to the identity. For the general situation where the electric field changes direction, the nonadiabatic factor can be deduced only by first separating out the geometric operator in the time evolution. This includes the cases studied in the previous section and in this section.

8 Appendix A

To derive the factorization of $U(t, 0)$, first switch to the Heisenberg picture. The equations of motion are

$$\dot{\pi} = -i\omega\pi + i\frac{qB}{c}\dot{R}(t), \quad \dot{w}_\mu = -\frac{qB}{c}\epsilon_{\mu\nu}\dot{R}_\nu(t), \quad (22)$$

where $\pi = \pi_1 + i\pi_2$, and $R(t) = R_1(t) + iR_2(t)$. The solution to the Heisenberg equations can then be expressed as

$$\pi(t) = \pi(0)e^{-i\omega t} + i\frac{qB}{c}e^{-i\omega t} \int_0^t e^{i\omega s} \frac{d}{ds} R(s) ds, \quad (23)$$

$$w_\mu(t) = w_\mu(0) - \frac{qB}{c}\epsilon_{\mu\nu}R_\nu(t), \quad (24)$$

where we assume $R_\nu(0) = 0$. The homogeneous terms in the expressions for $\pi(t)$ and $w_\mu(t)$ are generated by the usual dynamical operator $D(t) = \exp(-iH_0t/\hbar)$. To produce the extra terms in the expression for $\pi(t)$, and $w_\mu(t)$, respectively, using an operator $W(t)$, such that $D(t)W(t)$ recovers the whole solution, it suffices for $W(t)$ to satisfy:

$$W^\dagger(t)\pi(0)W(t) = \pi(0) + i\frac{qB}{c} \int_0^t e^{i\omega s} \frac{d}{ds} R(s) ds,$$

$$W^\dagger(t)w_\mu(0)W(t) = w_\mu(0) - \frac{qB}{c}\epsilon_{\mu\nu}(R_\nu(t) - R_\nu(0)).$$

In view of the commutation relations (1), which imply $[\pi, \pi^\dagger] = 2\hbar qB/c$, and from the formula $\exp(-B)A\exp(B) = A + [A, B]$ with the condition that $[A, B]$ commutes with A and B , it is clear that $W(t)$ can be chosen to be the product of two mutually commuting operators, generated by $(1, \pi(0), \pi^\dagger(0))$ and $(1, w_1(0), w_2(0))$ respectively. Each of these operators produces a translation for either $\pi(0)$ or $w_\mu(0)$ while leaving the other unchanged. Writing $W(t)$ as $W(t) = J(t)M(t)$, we can make the following choice for $J(t)$ and $M(t)$,

$$J(t) = T \exp \left(i\frac{\pi^\dagger(0)}{2\hbar} \int_0^t e^{i\omega s} \frac{d}{ds} R(s) ds + i\frac{\pi(0)}{2\hbar} \int_0^t e^{-i\omega s} \frac{d}{ds} R^*(s) ds \right), \quad (25)$$

$$M(t) = P \exp \left(-i\hbar^{-1}w_\mu(0)R_\mu(t) \right), \quad (26)$$

where $T \exp$ stands for time-ordered exponential. It's different from the direct exponential by a numerical phase factor only, similar to the path-ordered exponential. Therefore, it can be directly checked that $D(t)J(t)M(t)$ recovers the solutions to the Heisenberg equations.

To verify that $D(t)J(t)M(t)$ not only recovers the solutions to the Heisenberg equations for π and w_μ , but in fact is the time evolution operator corresponding to H , we now verify that it satisfies the Schrödinger equation. Note that $M(t)$ commutes with both $D(t)$ and $J(t)$, so we have

$$i\hbar \frac{\partial}{\partial t} (D(t)J(t)M(t)) = i\hbar \left[\frac{\partial}{\partial t} (D(t)M(t)) \right] J(t) + i\hbar M(t) D(t) \frac{\partial}{\partial t} J(t).$$

It is straightforward that

$$i\hbar\left[\frac{\partial}{\partial t}(D(t)M(t))\right]J(t) = (H_0(0) + w_\mu\dot{R}_\mu(t))D(t)J(t)M(t).$$

To calculate $i\hbar M(t)D(t)\frac{\partial}{\partial t}J(t)$, first observe that

$$D(t)\pi(0)D^\dagger(t) = D^\dagger(-t)\pi(0)D(-t) = \pi(0)e^{i\omega t}, \quad (27)$$

$$D(t)\pi^\dagger(0)D^\dagger(t) = (D(t)\pi(0)D^\dagger(t))^\dagger = \pi^\dagger(0)e^{-i\omega t}. \quad (28)$$

Therefore

$$i\hbar M(t)D(t)\frac{\partial}{\partial t}J(t) = (-\pi^\dagger(0)\dot{R}(t)/2 - \pi(0)\dot{R}^*(t)/2)D(t)J(t)M(t), \quad (29)$$

$$= -(\pi_1(0)\dot{R}_1(t) + \pi_2(0)\dot{R}_2(t))D(t)J(t)M(t). \quad (30)$$

Combining terms and from the definitions of π_μ and w_μ , we now have

$$i\hbar\frac{\partial}{\partial t}(D(t)J(t)M(t)) = (H_0 + \frac{qB}{c}x_1\dot{R}_2 - \frac{qB}{c}x_2\dot{R}_1)D(t)J(t)M(t), \quad (31)$$

$$= H(D(t)J(t)M(t)). \quad (32)$$

Therefore, we conclude that the time evolution operator corresponding to H is

$$U(t, 0) = D(t)J(t)M(t) = M(t)D(t)J(t). \quad (33)$$

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